Průběh funkce

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Obsah

Vyšetřete chování funkce $y = \dfrac{x}{1 + x^2}$

$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R};$

- Omezení na definiční obor vyplývá ze jmenovatele zlomku.
- Výraz $x^2 + 1$ nesmí být nulový.
- To je však zajištěno pro všechna reálná čísla.

$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

- Čitatel, x, je lichá funkce, jmenovatel, $(1 + x^2)$, je funkce sudá.
- Jako celek je tedy zlomek lichá funkce.

$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

$y = 0$

Určíme průsečík s osou x a znaménko funkce na jednotlivých intervalech.

$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0$$

Určíme průsečík s osou x a znaménko funkce na jednotlivých intervalech.

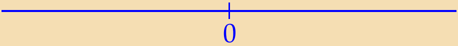
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$

Zlomek je roven nule právě tehdy, když čitatel je nulový.

$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

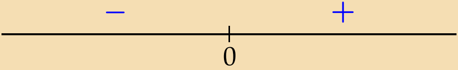
$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$



Zakreslíme průsečík $x = 0$ na osu x. Funkce nemá žádný bod nespojitosti.

$\boxed{y=\dfrac{x}{1+x^2}}$ $D(f)=\mathbb{R}$; lichá;

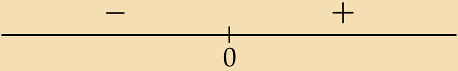
$$y=0 \Rightarrow \frac{x}{1+x^2}=0 \Rightarrow x=0$$



- Jmenovatel $(1+x^2)$ je stále kladný.
- Čitatel zlomku má proto stejné znaménko jako celý zlomek $\dfrac{x}{1+x^2}$.
- Funkce je kladná, je-li x kladné a naopak.

$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$

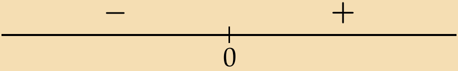


$$\lim_{x\to\pm\infty} \frac{x}{1+x^2}$$

Určíme limity v nekonečnu.

$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$

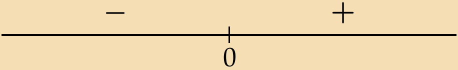


$$\lim_{x\to\pm\infty} \frac{x}{1+x^2} = \lim_{x\to\pm\infty} \frac{1}{x}$$

- Víme, že o výsledku rozhodují jenom vedoucí členy v čitateli a ve jmenovateli.
- Zelenou část lze vynechat.
- Zbytek zkrátíme: $\frac{x}{x^2} = \frac{1}{x}$.

$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$

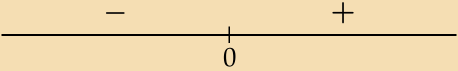


$$\lim_{x\to\pm\infty} \frac{x}{1+x^2} = \lim_{x\to\pm\infty} \frac{1}{x} = \frac{1}{\pm\infty}$$

Dosadíme.

$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

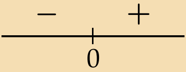
$$y = 0 \Rightarrow \frac{x}{1+x^2} = 0 \Rightarrow x = 0$$



$$\lim_{x\to\pm\infty} \frac{x}{1+x^2} = \lim_{x\to\pm\infty} \frac{1}{x} = \frac{1}{\pm\infty} = 0$$

- Obě hodnoty $\frac{1}{\infty}$ i $\frac{1}{-\infty}$ jsou nulové.
- Funkce má vodorovnou asymptotu $y = 0$ pro x jdoucí k $\pm\infty$.

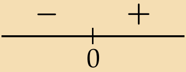
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1(1+x^2) - x(0+2x)}{(1+x^2)^2}$$

- Vypočteme derivaci.
- Derivujeme podíl podle vzorce pro derivaci podílu.

$\boxed{y=\frac{x}{1+x^2}}$ $D(f)=\mathbb{R}$; lichá;

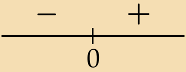


$$\begin{aligned}y' &= \frac{1(1+x^2)-x(0+2x)}{(1+x^2)^2}\\ &= \frac{1+x^2-2x^2}{(1+x^2)^2}\end{aligned}$$

Upravíme.

$$\boxed{y = \frac{x}{1+x^2}}$$

$D(f) = \mathbb{R}$; lichá;



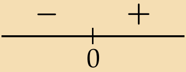
$$y' = \frac{1(1+x^2) - x(0+2x)}{(1+x^2)^2}$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

Upravíme.

$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;

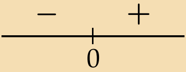


$$y' = \frac{1-x^2}{(1+x^2)^2},$$

$$y' = 0$$

Hledáme řešení rovnice $y' = 0$.

$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



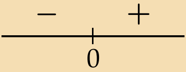
$$y' = \frac{1-x^2}{(1+x^2)^2},$$

$$y' = 0$$

$$\frac{1-x^2}{(1+x^2)^2} = 0$$

Dosadíme za derivaci.

$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;

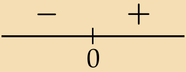


$$y' = \frac{1-x^2}{(1+x^2)^2},$$

$$\begin{aligned} y' &= 0 \\ \frac{1-x^2}{(1+x^2)^2} &= 0 \\ 1-x^2 &= 0 \end{aligned}$$

Zlomek je nulový, má-li nulový čitatel.

$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;

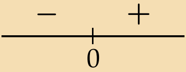


$$y' = \frac{1-x^2}{(1+x^2)^2},$$

$$\begin{aligned} y' &= 0 \\ \frac{1-x^2}{(1+x^2)^2} &= 0 \\ 1-x^2 &= 0 \\ x^2 &= 1 \end{aligned}$$

Vyjádříme x^2.

$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



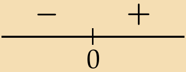
$$y' = \frac{1-x^2}{(1+x^2)^2},$$

$$\begin{aligned} y' &= 0 \\ \frac{1-x^2}{(1+x^2)^2} &= 0 \\ 1-x^2 &= 0 \\ x^2 &= 1 \\ x_1 &= 1 \\ x_2 &= -1 \end{aligned}$$

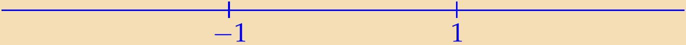
Vypočítáme x. Dostáváme dvě řešení.

$$\boxed{y = \frac{x}{1+x^2}}$$

$D(f) = \mathbb{R}$; lichá;

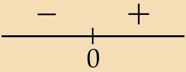


$$y' = \frac{1-x^2}{(1+x^2)^2}, \qquad x_{1,2} = \pm 1$$

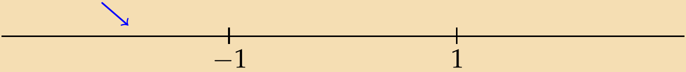


- Nakreslíme osu x a stacionární body.
- Nejsou žádné body nespojitosti.

$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;



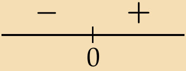
$$y' = \frac{1-x^2}{(1+x^2)^2}, \quad x_{1,2} = \pm 1$$



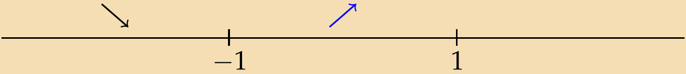
Testujeme $x = -2$. Dostáváme

$$y'(-2) = \frac{1-4}{\text{kladná hodnota}} < 0.$$

$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



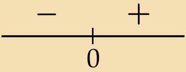
$y' = \dfrac{1-x^2}{(1+x^2)^2}$, $\quad x_{1,2} = \pm 1$



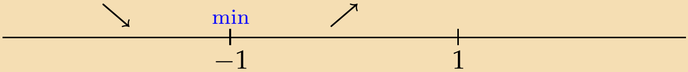
Testujeme $x = 0$.

$$y'(0) = \frac{1}{1} > 0$$

$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



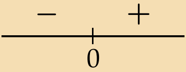
$$y' = \frac{1-x^2}{(1+x^2)^2}, \quad x_{1,2} = \pm 1$$



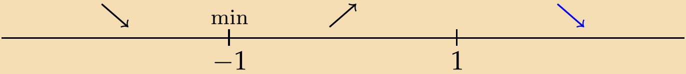
Funkce má lokální minimum v bodě $x = -1$. Funkční hodnota je

$$y(-1) = \frac{-1}{1+(-1)^2} = -\frac{1}{2}.$$

$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;



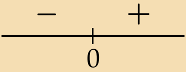
$$y' = \frac{1-x^2}{(1+x^2)^2}, \quad x_{1,2} = \pm 1$$



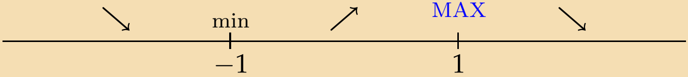
Testujeme $x = 2$. Platí

$$y'(2) = \frac{1-4}{\text{kladná hodnota}} < 0.$$

$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}, \quad x_{1,2} = \pm 1$$

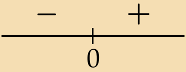


Funkce má lokální maximum v bodě $x = 1$. Funkční hodnota je

$$y(1) = -y(-1) = \frac{1}{2},$$

kde jsme využili toho, že funkce je lichá a hodnota $y(-1)$ již byla vypočítána.

$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

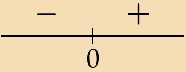


$$y' = \frac{1-x^2}{(1+x^2)^2}, \quad x_{1,2} = \pm 1$$

$$y'' = \left(\frac{1-x^2}{(1+x^2)^2}\right)'$$

Vypočteme druhou derivaci.

$\boxed{y=\dfrac{x}{1+x^2}}$ $D(f)=\mathbb{R}$; lichá;

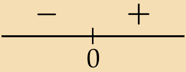


$$y'=\frac{1-x^2}{(1+x^2)^2},\quad x_{1,2}=\pm 1$$

$$y''=\left(\frac{1-x^2}{(1+x^2)^2}\right)'$$
$$=\frac{-2x(1+x^2)^2-(1-x^2)2(1+x^2)(0+2x)}{(1+x^2)^4}$$

- Derivuje podíl podle vzorce pro derivaci podílu.
- Jmenovatel derivujeme jako složenou funkci. Tím se nezbavíme možnosti vytknout v čitateli a zkrátit zlomek.

$\boxed{y=\frac{x}{1+x^2}}$ $D(f)=\mathbb{R}$; lichá;

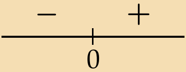


$$y'=\frac{1-x^2}{(1+x^2)^2}, \quad x_{1,2}=\pm 1$$

$$\begin{aligned}
y'' &= \left(\frac{1-x^2}{(1+x^2)^2}\right)' \\
&= \frac{-2x(1+x^2)^2-(1-x^2)2(1+x^2)(0+2x)}{(1+x^2)^4} \\
&= \frac{-2x(1+x^2)[(1+x^2)+(1-x^2)2]}{(1+x^2)^4}
\end{aligned}$$

Vytkneme

$\boxed{y=\frac{x}{1+x^2}}$ $D(f)=\mathbb{R}$; lichá;

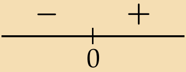


$$y'=\frac{1-x^2}{(1+x^2)^2},\qquad x_{1,2}=\pm 1$$

$$\begin{aligned}
y''&=\left(\frac{1-x^2}{(1+x^2)^2}\right)'\\
&=\frac{-2x(1+x^2)^2-(1-x^2)2(1+x^2)(0+2x)}{(1+x^2)^4}\\
&=\frac{-2x(1+x^2)[\ 1+x^2\ \ +(1-x^2)2]}{(1+x^2)^4}\\
&=\frac{-2x[3-x^2]}{(1+x^2)^3}
\end{aligned}$$

Zelené části se zkrátí. Zjednodušíme výraz v hranaté závorce.

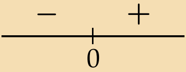
$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}, \qquad x_{1,2} = \pm 1$$

$$\begin{aligned}
y'' &= \left(\frac{1-x^2}{(1+x^2)^2}\right)' \\
&= \frac{-2x(1+x^2)^2 - (1-x^2)2(1+x^2)(0+2x)}{(1+x^2)^4} \\
&= \frac{-2x(1+x^2)[\ 1+x^2\ \ +(1-x^2)2]}{(1+x^2)^4} \\
&= \frac{-2x[3-x^2]}{(1+x^2)^3} \\
&= 2\frac{x(x^2-3)}{(1+x^2)^3}
\end{aligned}$$

$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

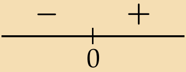


$$y' = \frac{1-x^2}{(1+x^2)^2}, \quad x_{1,2} = \pm 1$$

$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0$$

Vyřešíme $y'' = 0$.

$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

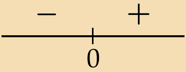


$$y' = \frac{1-x^2}{(1+x^2)^2}, \qquad x_{1,2} = \pm 1$$

$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

Zlomek je nulový, je-li nulový jeho čitatel.

$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}, \qquad x_{1,2} = \pm 1$$

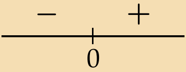
$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$

Jsou dvě možnosti: buď $x = 0$, nebo $x^2 - 3 = 0$. Druhá z možností vede na rovnici

$$x^2 = 3$$
$$x = \pm\sqrt{3}.$$

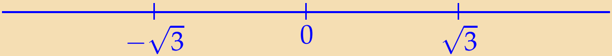
$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;



$$y' = \frac{1-x^2}{(1+x^2)^2}, \qquad x_{1,2} = \pm 1$$

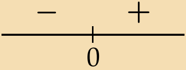
$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$

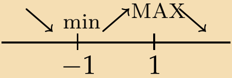


Vyznačíme body na osu x. Nejsou zde žádné body nespojitosti.

$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

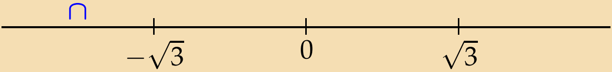


$$y' = \frac{1-x^2}{(1+x^2)^2}, \quad x_{1,2} = \pm 1$$



$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

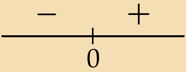
$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$



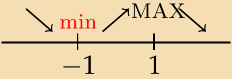
Testujeme $x = -2$.

$$y''(-2) = 2\frac{-2(4-3)}{\text{kladná hodnota}} < 0.$$

$\boxed{y = \frac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

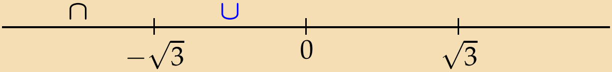


$$y' = \frac{1-x^2}{(1+x^2)^2}, \quad x_{1,2} = \pm 1$$



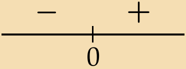
$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$

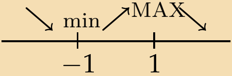


Testujeme $x = -1$. Funkce je v tomto bodě konvexní, protože je zde lokální minimum.

$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

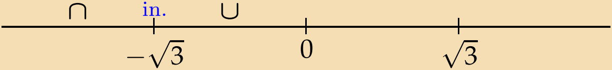


$$y' = \frac{1-x^2}{(1+x^2)^2}, \quad x_{1,2} = \pm 1$$



$$\color{red}{y'' = 2\frac{x(x^2-3)}{(1+x^2)^3}} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

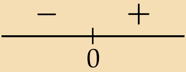
$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$



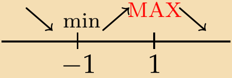
V bodě $x = -\sqrt{3}$ je inflexe. Funkční hodnota je

$$y(-\sqrt{3}) = \frac{-\sqrt{3}}{1+3} \approx -0.43.$$

$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;

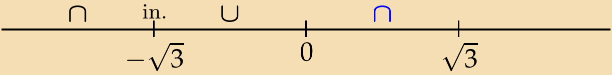


$$y' = \frac{1-x^2}{(1+x^2)^2}, \quad x_{1,2} = \pm 1$$



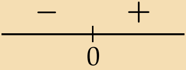
$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

$$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$$

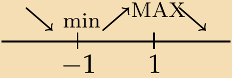


Testujeme $x = 1$. Funkce je v tomto bodě konkávní, protože je zde lokální maximum.

$$\boxed{y = \frac{x}{1+x^2}}$$ $D(f) = \mathbb{R}$; lichá;

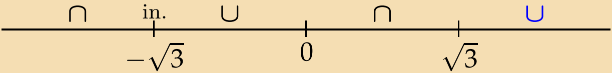


$$y' = \frac{1-x^2}{(1+x^2)^2}, \quad x_{1,2} = \pm 1$$



$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

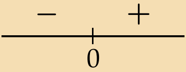
$$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$$



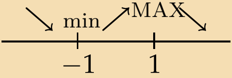
Testujeme $x = 2$. Dostáváme

$$y''(2) = 2\frac{2(4-3)}{\text{něco kladného}} > 0.$$

$\boxed{y = \dfrac{x}{1+x^2}}$ $D(f) = \mathbb{R}$; lichá;

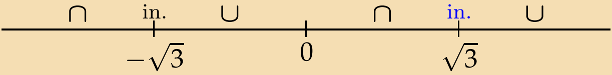


$y' = \dfrac{1-x^2}{(1+x^2)^2}$, $\quad x_{1,2} = \pm 1$



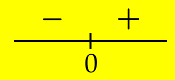
$$y'' = 2\frac{x(x^2-3)}{(1+x^2)^3} \quad \Rightarrow \quad 2\frac{x(x^2-3)}{(1+x^2)^3} = 0 \quad \Rightarrow \quad x(x^2-3) = 0$$

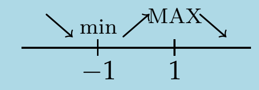
$x_3 = 0,\ x_4 = \sqrt{3},\ x_5 = -\sqrt{3}$

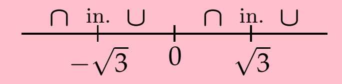


Inflexe v bodě $x = \sqrt{3}$. Funkční hodnota je

$$y(\sqrt{3}) = \frac{\sqrt{3}}{1+3} \approx 0.43.$$





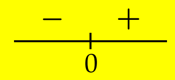


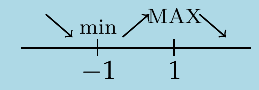
$f(0) = 0$
$f(\pm\infty) = 0$

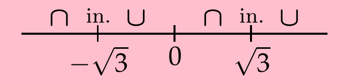
$$f(\pm 1) = \pm\frac{1}{2}$$

$$f(\pm\sqrt{3}) \approx \pm 0.433$$

Vypíšeme si nejdůležitější výsledky.







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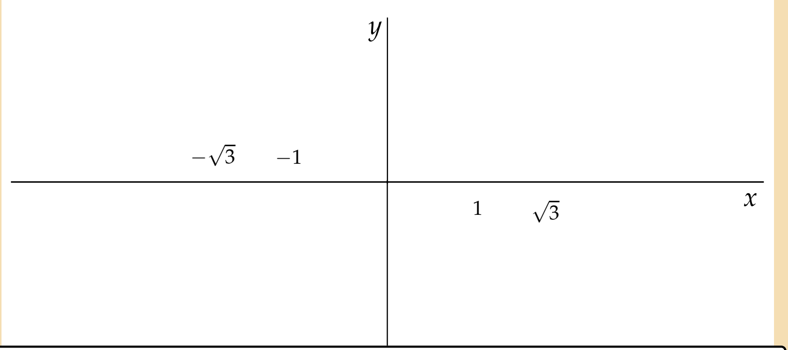
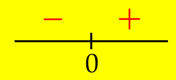
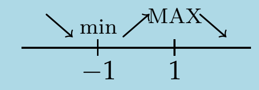
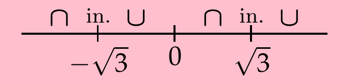


Zakreslíme souřadný systém.







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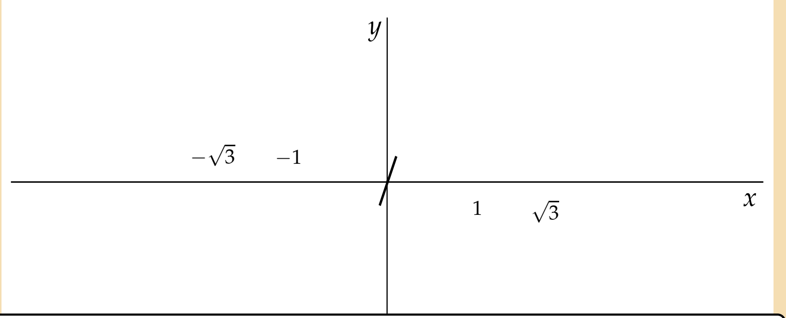
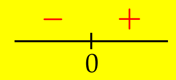
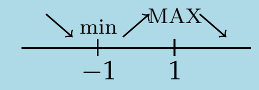
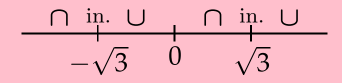


V bodě $x = 0$ je průsečík s osou x. Funkční hodnoty se v tomto bodě mění z kladných na záporné.



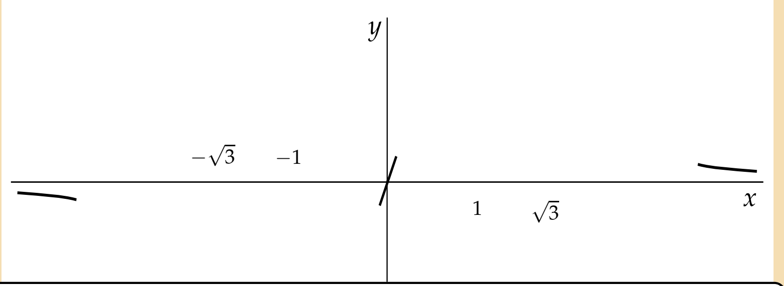




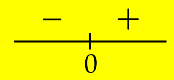
$f(0) = 0$
$f(\pm\infty) = 0$

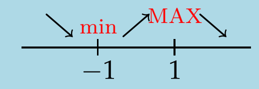
$$f(\pm 1) = \pm\frac{1}{2}$$

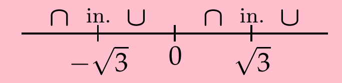
$$f(\pm\sqrt{3}) \approx \pm 0.433$$



Zachytíme informaci o vodorovné tečně v ±∞. Dáváme si pozor na znaménko funkce, musíme graf správně nakreslit nad nebo pod asymptotu.







$f(0) = 0$
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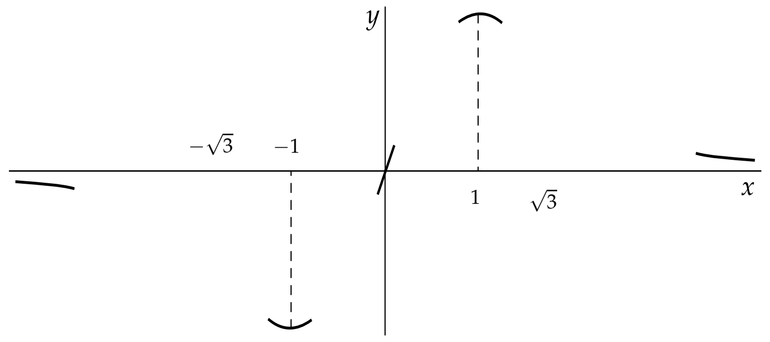
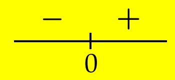
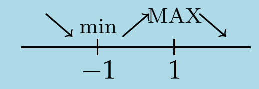




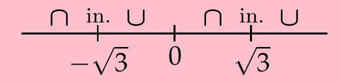
$-$ $+$

0



↘ min ↗ MAX ↘

-1 1



$\cap$ in. $\cup$ $\cap$ in. $\cup$

$-\sqrt{3}$ 0 $\sqrt{3}$

$f(0) = 0$
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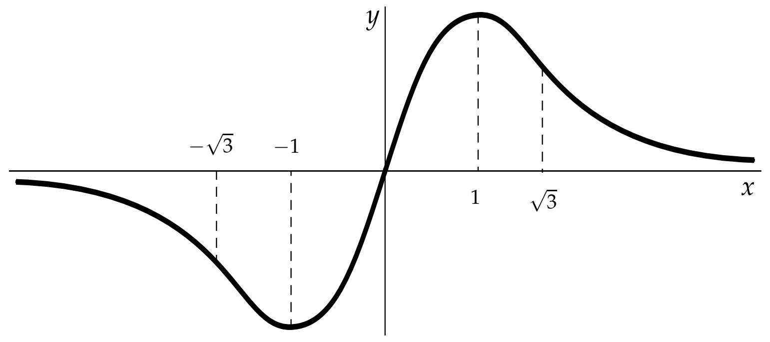


Vyšetřete chování funkce $y = \dfrac{3x+1}{x^3}$

$$\boxed{y = \frac{3x+1}{x^3}}$$

$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;

- Určíme definiční obor.
- Ve jmenovateli nesmí být nula.

$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;

$$y = 0$$

Určíme průsečík s osou x jako řešení rovnice $y = 0$.

$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;

$$y = 0$$

$$\frac{3x+1}{x^3} = 0$$

$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;

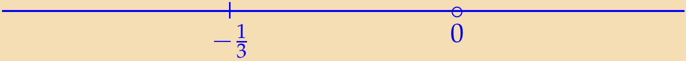
$$\begin{gathered} y = 0 \\ \frac{3x+1}{x^3} = 0 \\ 3x + 1 = 0 \end{gathered}$$

$\boxed{y=\frac{3x+1}{x^3}}$ $D(f)=\mathbb{R}\setminus\{0\}$;

$$\begin{aligned} y&=0\\ \frac{3x+1}{x^3}&=0\\ 3x+1&=0\\ x&=-\frac{1}{3}\end{aligned}$$

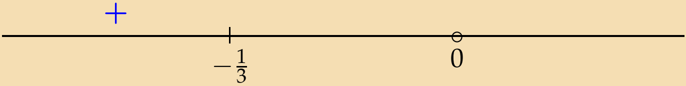
Funkce má s osou x jediný průsečík $x=-\dfrac{1}{3}$

$\boxed{y = \dfrac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



- Určíme znaménka funkce.
- Rozdělíme osu x pomocí průsečíků a bodů nespojitosti na podintervaly, kde se znaménko zachovává.

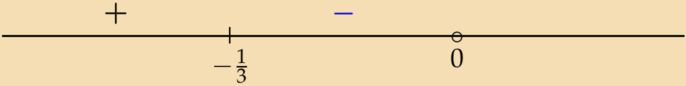
$\boxed{y = \dfrac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



Uvažujme interval zcela vlevo. Zvolme $x = -1$ a vypočteme

$$y(-1) = \frac{-3+1}{-1} = 2 > 0.$$

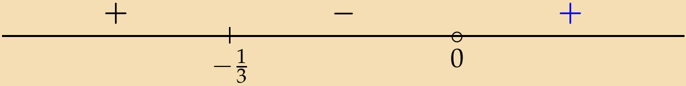
$$\boxed{y = \frac{3x+1}{x^3}}\quad D(f) = \mathbb{R} \setminus \{0\}\ ;$$



Uvažujme prostřední interval, zvolme $x = -\dfrac{1}{4}$ a vypočteme

$$y(-\frac{1}{4}) = \frac{-\frac{3}{4}+1}{-\frac{1}{64}} = \frac{\frac{1}{4}}{-\frac{1}{64}} = -16 < 0.$$

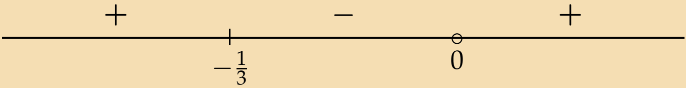
$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



V posledním intervalu zvolme $x = 1$ a vypočteme

$$y(1) = \frac{3+1}{1} = 4 > 0.$$

$$\boxed{y = \frac{3x+1}{x^3}} \quad D(f) = \mathbb{R} \setminus \{0\}\ ;$$

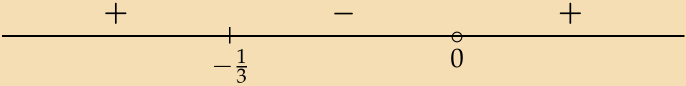


$$\lim_{x\to 0^+} \frac{3x+1}{x^3} =$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} =$$

Najdeme jednostranné limity v bodech nespojitosti.

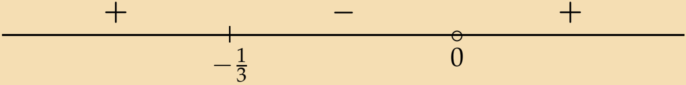
$$\boxed{y = \frac{3x+1}{x^3}} \quad D(f) = \mathbb{R} \setminus \{0\}\ ;$$



$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{0}$$
$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{0}$$

Dosazení $x = 0$ vede k výrazu typu $\frac{\text{nenulový výraz}}{\text{nula}}$.

$$\boxed{y = \frac{3x+1}{x^3}} \quad D(f) = \mathbb{R} \setminus \{0\}\ ;$$

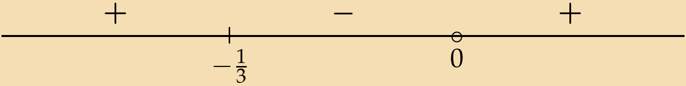


$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$

- Z přednášky víme, že jednostranné limity jsou nevlastní.
- Schéma se znaménkem funkce umožňuje odhalit, zda se funkce blíží k plus nebo minus nekonečnu.

$$\boxed{y = \frac{3x+1}{x^3}}\quad D(f) = \mathbb{R} \setminus \{0\}\ ;$$



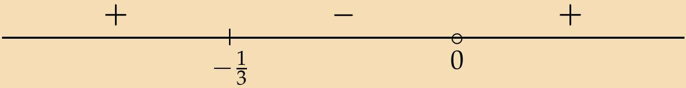
$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$

$$\lim_{x\to \pm\infty} \frac{3x+1}{x^3}$$

Určíme limity v nevlastních bodech.

$$\boxed{y=\frac{3x+1}{x^3}}\quad D(f)=\mathbb{R}\setminus\{0\}\ ;$$



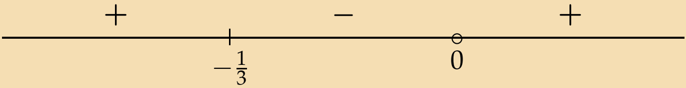
$$\lim_{x\to 0^+}\frac{3x+1}{x^3}=\frac{1}{+0}=\infty$$

$$\lim_{x\to 0^-}\frac{3x+1}{x^3}=\frac{1}{-0}=-\infty$$

$$\lim_{x\to\pm\infty}\frac{3x+1}{x^3}$$

Víme, že pouze vedoucí členy jsou podstatné v limitě tohoto typu a ostatní členy můžeme vynechat.

$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



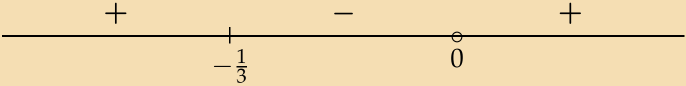
$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$

$$\lim_{x\to \pm\infty} \frac{3x+1}{x^3} = \lim_{x\to \pm\infty} \frac{3}{x^2}$$

Zkrátíme x.

$$\boxed{y = \frac{3x+1}{x^3}} \quad D(f) = \mathbb{R} \setminus \{0\}\ ;$$



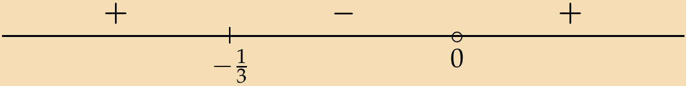
$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$

$$\lim_{x\to \pm\infty} \frac{3x+1}{x^3} = \lim_{x\to \pm\infty} \frac{3}{x^2} = \frac{3}{\infty}$$

Dosadíme.

$\boxed{y = \dfrac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



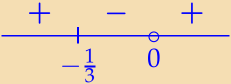
$$\lim_{x\to 0^+} \frac{3x+1}{x^3} = \frac{1}{+0} = \infty$$

$$\lim_{x\to 0^-} \frac{3x+1}{x^3} = \frac{1}{-0} = -\infty$$

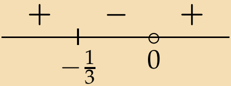
$$\lim_{x\to \pm\infty} \frac{3x+1}{x^3} = \lim_{x\to \pm\infty} \frac{3}{x^2} = \frac{3}{\infty} = 0$$

Limita ja vypočtena.
Funkce má vodorovnou asymptotu $y = 0$ v $\pm\infty$.

$\boxed{y=\frac{3x+1}{x^3}}$ $D(f)=\mathbb{R}\setminus\{0\}$;



$$\boxed{y=\frac{3x+1}{x^3}}\quad D(f)=\mathbb{R}\setminus\{0\}\ ;$$

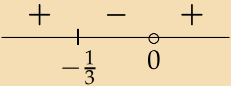


$$y'=\frac{3x^3-(3x+1)3x^2}{(x^3)^2}$$

Derivujeme podíl.

$$\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$$

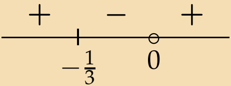
$$\boxed{y=\frac{3x+1}{x^3}}\quad D(f)=\mathbb{R}\setminus\{0\}\,;$$



$$y'=\frac{3x^3-(3x+1)3x^2}{(x^3)^2}=\frac{3x^2\Big(x-(3x+1)\Big)}{x^6}$$

Vytknutím rozložíme na součin.

$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;

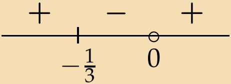


$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2} = \frac{3x^2\Big(x - (3x+1)\Big)}{x^6}$$

$$= 3\frac{x - 3x - 1}{x^4}$$

- Zkrátíme.
- Roznásobíme závorku.

$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;

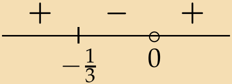


$$y' = \frac{3x^3 - (3x+1)3x^2}{(x^3)^2} = \frac{3x^2\Big(x - (3x+1)\Big)}{x^6}$$

$$= 3\frac{x - 3x - 1}{x^4} = 3\frac{-2x-1}{x^4}$$

Zjednodušíme.

$$\boxed{y=\frac{3x+1}{x^3}}\quad D(f)=\mathbb{R}\setminus\{0\}\ ;$$

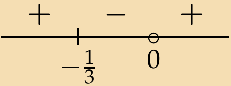


$$y'=\frac{3x^3-(3x+1)3x^2}{(x^3)^2}=\frac{3x^2\Big(x-(3x+1)\Big)}{x^6}$$

$$=3\frac{x-3x-1}{x^4}=3\frac{-2x-1}{x^4}=-3\frac{2x+1}{x^4}$$

Máme derivaci.

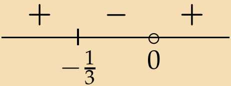
$\boxed{y = \dfrac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



$y'(x) = -3\dfrac{2x+1}{x^4}$;

Máme derivaci.

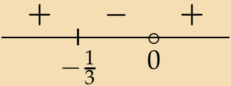
$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;



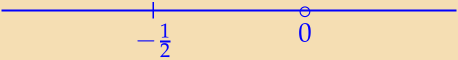
$$y'(x) = -3\frac{2x+1}{x^4}\ ;\ x_1 = -\frac{1}{2}$$

Rovnice $y' = 0$ je ekvivalentní rovnici $2x + 1 = 0$.

$\boxed{y=\frac{3x+1}{x^3}}$ $D(f)=\mathbb{R}\setminus\{0\}$;



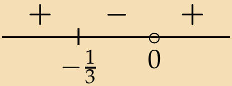
$y'(x)=-3\frac{2x+1}{x^4}$; $x_1=-\frac{1}{2}$



Vyznačíme stacionární bod a bod nespojitosti na osu x.

$$y = \frac{3x+1}{x^3}$$

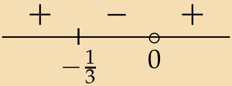
$D(f) = \mathbb{R} \setminus \{0\}$;



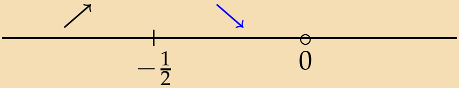
$y'(x) = -3\frac{2x+1}{x^4}$; $x_1 = -\frac{1}{2}$

$$y'(-1) = -3\frac{-2+1}{1} = 3 > 0$$

$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;

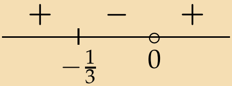


$$y'(x) = -3\frac{2x+1}{x^4}\ ;\ x_1 = -\frac{1}{2}$$

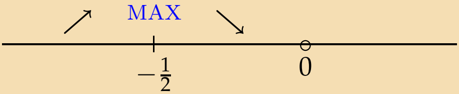


$y'(-\frac{1}{3}) < 0$, protože funkce mění znaménko z kladného na záporné.

$$\boxed{y = \frac{3x+1}{x^3}}\quad D(f) = \mathbb{R}\setminus\{0\}\ ;$$



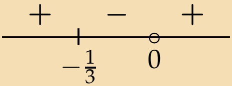
$$y'(x) = -3\frac{2x+1}{x^4}\ ;\ x_1 = -\frac{1}{2}$$



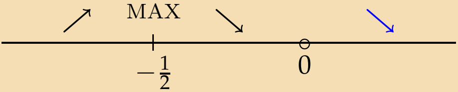
Funkce má lokální minimum v bodě $x = -\dfrac{1}{2}$. Funkční hodnota je

$$y(-\frac{1}{2}) = \frac{-\frac{3}{2}+1}{-\frac{1}{8}} = \frac{-\frac{1}{2}}{-\frac{1}{8}} = 4.$$

$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



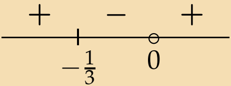
$y'(x) = -3\frac{2x+1}{x^4}$; $x_1 = -\frac{1}{2}$



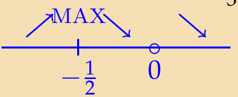
$$\boxed{y'(1) = -3\frac{3}{1} = -9 > 0}$$

$$\boxed{y = \frac{3x+1}{x^3}}$$

$D(f) = \mathbb{R} \setminus \{0\}$;

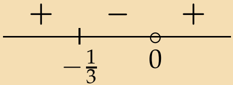


$$y'(x) = -3\frac{2x+1}{x^4} \ ;$$

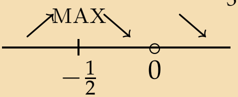


$$y'' = -3\left(\frac{2x+1}{x^4}\right)'$$

$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;

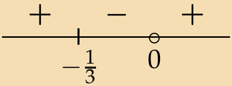


$y'(x) = -3\frac{2x+1}{x^4}$;

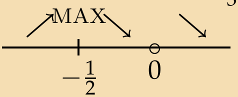


$$y'' = -3\left(\frac{2x+1}{x^4}\right)' = -3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$

$\boxed{y=\dfrac{3x+1}{x^3}}$ $D(f)=\mathbb{R}\setminus\{0\}$;



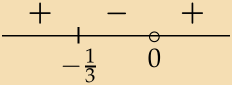
$y'(x)=-3\dfrac{2x+1}{x^4}$;



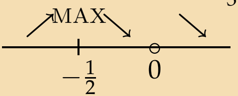
$$y''=-3\left(\frac{2x+1}{x^4}\right)'=-3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$

$$=-3\frac{2x^4-8x^4-4x^3}{x^8}$$

$$\boxed{y=\frac{3x+1}{x^3}}\quad D(f)=\mathbb{R}\setminus\{0\}\ ;$$



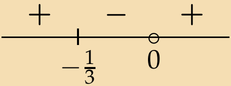
$$y'(x)=-3\frac{2x+1}{x^4}\ ;$$



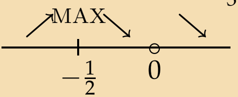
$$y''=-3\left(\frac{2x+1}{x^4}\right)'=-3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$

$$=-3\frac{2x^4-8x^4-4x^3}{x^8}=-3\frac{-6x^4-4x^3}{x^8}$$

$$\boxed{y = \frac{3x+1}{x^3}} \quad D(f) = \mathbb{R} \setminus \{0\}\ ;$$



$$y'(x) = -3\frac{2x+1}{x^4}\ ;$$

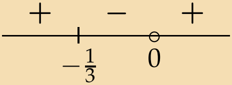


$$y'' = -3\left(\frac{2x+1}{x^4}\right)' = -3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$

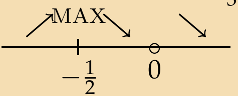
$$= -3\frac{2x^4-8x^4-4x^3}{x^8} = -3\frac{-6x^4-4x^3}{x^8}$$

$$= 6\frac{3x^4+2x^3}{x^8}$$

$\boxed{y=\frac{3x+1}{x^3}}$ $D(f)=\mathbb{R}\setminus\{0\}$;

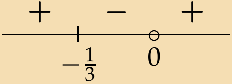


$y'(x)=-3\frac{2x+1}{x^4}$;

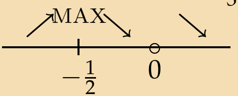


$$
\begin{aligned}
y'' &= -3\left(\frac{2x+1}{x^4}\right)' = -3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}\\
&= -3\frac{2x^4-8x^4-4x^3}{x^8} = -3\frac{-6x^4-4x^3}{x^8}\\
&= 6\frac{3x^4+2x^3}{x^8} = 6\frac{(3x+2)x^3}{x^8}
\end{aligned}
$$

$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;

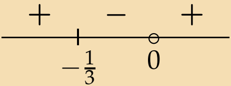


$y'(x) = -3\frac{2x+1}{x^4}$;

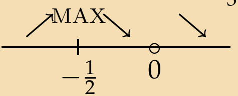


$$y'' = -3\left(\frac{2x+1}{x^4}\right)' = -3\frac{2x^4-(2x+1)4x^3}{(x^4)^2}$$
$$= -3\frac{2x^4-8x^4-4x^3}{x^8} = -3\frac{-6x^4-4x^3}{x^8}$$
$$= 6\frac{3x^4+2x^3}{x^8} = 6\frac{(3x+2)x^3}{x^8}$$
$$= 6\frac{3x+2}{x^5}$$

$$\boxed{y = \frac{3x+1}{x^3}} \quad D(f) = \mathbb{R} \setminus \{0\}\ ;$$

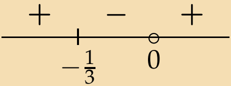


$$y'(x) = -3\frac{2x+1}{x^4}\ ;$$

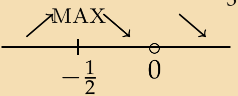


$$y'' = 6\frac{3x+2}{x^5};$$

$\boxed{y = \frac{3x+1}{x^3}}$ $D(f) = \mathbb{R} \setminus \{0\}$;



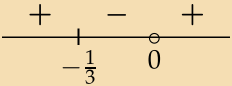
$y'(x) = -3\frac{2x+1}{x^4}$;



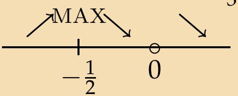
$y'' = 6\frac{3x+2}{x^5}$; $x_2 = -\frac{2}{3}$

$y'' = 0$ pro $3x + 2 = 0$, t.j. $x = -\frac{2}{3}$.

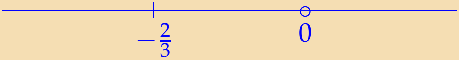
$\boxed{y=\frac{3x+1}{x^3}}$ $D(f)=\mathbb{R}\setminus\{0\}$;



$y'(x)=-3\frac{2x+1}{x^4}$;

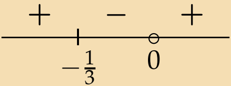


$y''=6\frac{3x+2}{x^5}$; $x_2=-\frac{2}{3}$



$$\boxed{y = \frac{3x+1}{x^3}}$$

$D(f) = \mathbb{R} \setminus \{0\}$;



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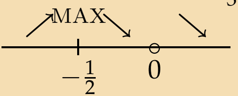
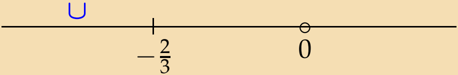
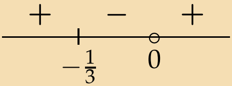


$y'' = 6\frac{3x+2}{x^5}$; $x_2 = -\frac{2}{3}$

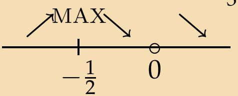


$$\boxed{y''(-1) = 6\frac{-1}{-1} = 6 > 0}$$

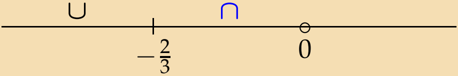
$$\boxed{y=\frac{3x+1}{x^3}}$$ $D(f)=\mathbb{R}\setminus\{0\}$;



$$y'(x)=-3\frac{2x+1}{x^4}\ ;$$

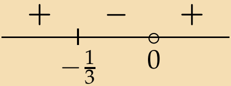


$$y''=6\frac{3x+2}{x^5};\ x_2=-\frac{2}{3}$$

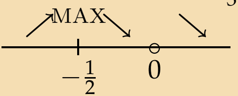


$$\boxed{y''(-\frac{1}{3})=6\frac{-1+2}{-\frac{1}{3^5}}<0}$$

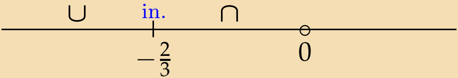
$$\boxed{y = \frac{3x+1}{x^3}}$$ $D(f) = \mathbb{R} \setminus \{0\}$;



$$y'(x) = -3\frac{2x+1}{x^4} \;;$$

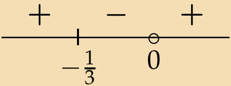


$$y'' = 6\frac{3x+2}{x^5}; \; x_2 = -\frac{2}{3}$$

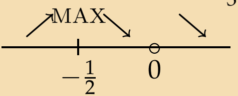


Inflexní bod $x = -\frac{2}{3}$. $\quad y(-\frac{2}{3}) = \dfrac{-2+1}{-\frac{2^5}{3^5}} \approx 3.375$

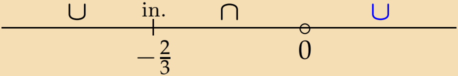
$$\boxed{y = \frac{3x+1}{x^3}} \quad D(f) = \mathbb{R} \setminus \{0\}\ ;$$



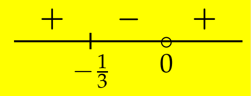
$$y'(x) = -3\frac{2x+1}{x^4}\ ;$$

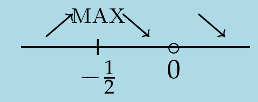


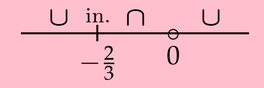
$$y'' = 6\frac{3x+2}{x^5};\ x_2 = -\frac{2}{3}$$



$$\boxed{y''(1) = 6\frac{5}{1} = 30 > 0}$$







$$f(-\frac{1}{3}) = 0$$

$$f(-\frac{1}{2}) = 4$$

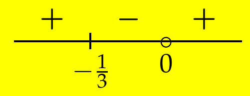
$$f(-\frac{2}{3}) \approx 3.4$$

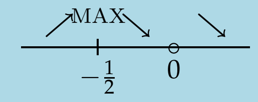
$$f(\pm\infty) = 0,$$

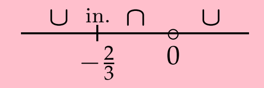
$$f(0+) = \infty,$$

$$f(0-) = -\infty$$

Shrneme dosažené výsledky.







$$f\left(-\frac{1}{3}\right) = 0$$

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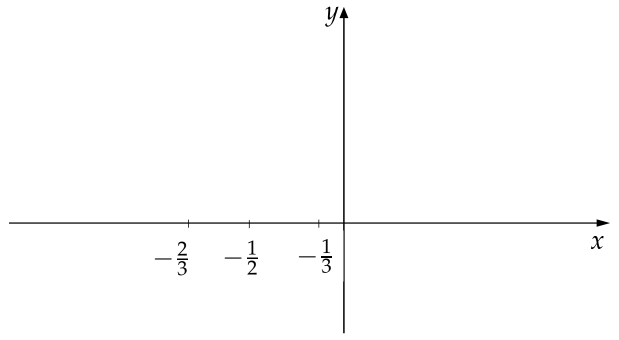
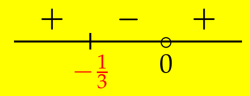
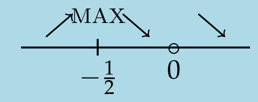




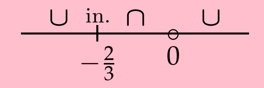
$+ \quad - \quad +$

$-\frac{1}{3} \quad 0$



$\nearrow$ MAX $\searrow \quad \searrow$

$-\frac{1}{2} \quad 0$



$\cup$ in. $\cap \quad \cup$

$-\frac{2}{3} \quad 0$

$f(-\frac{1}{3}) = 0$

$f(-\frac{1}{2}) = 4$

$f(-\frac{2}{3}) \approx 3.4$

$f(\pm\infty) = 0,$

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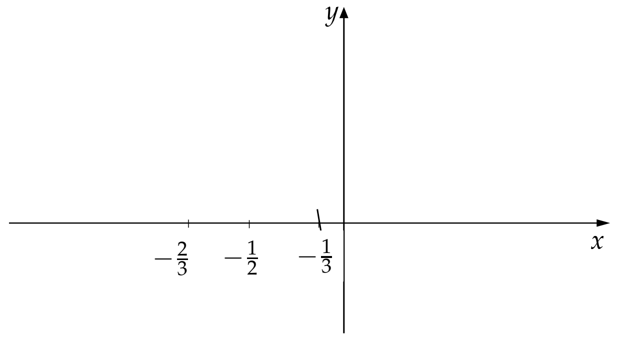

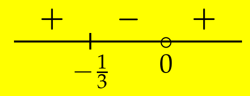

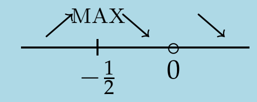

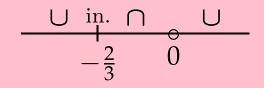

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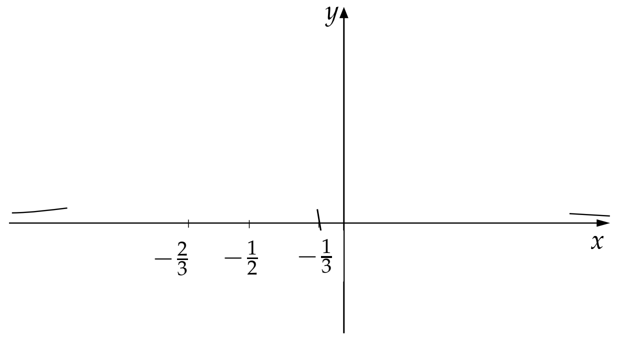

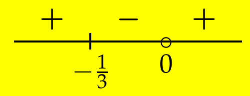

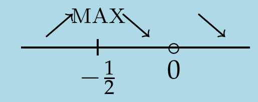

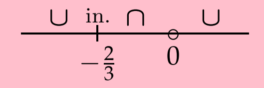

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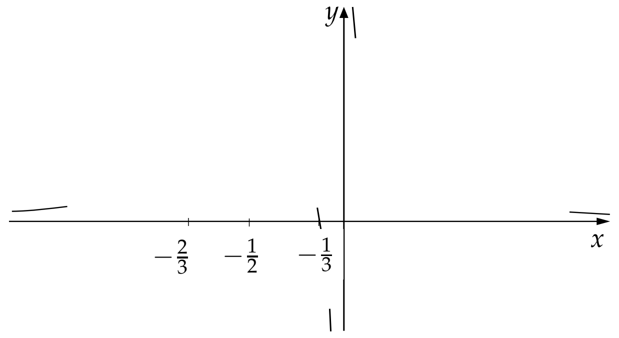

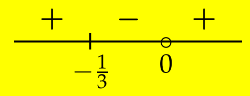

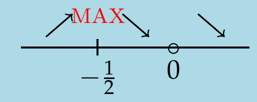

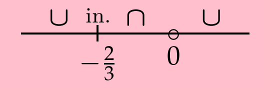

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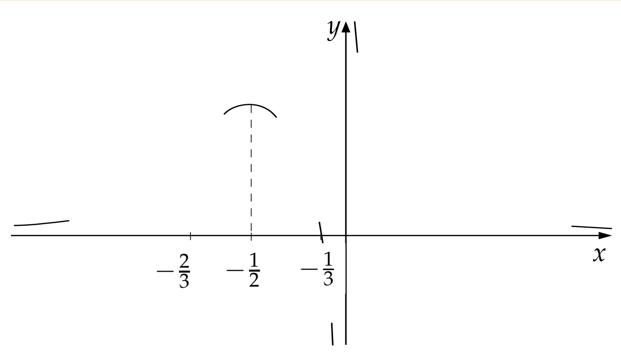

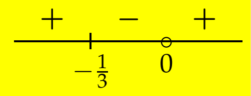

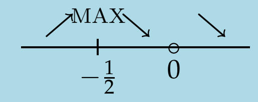

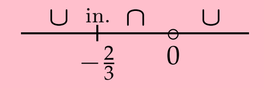

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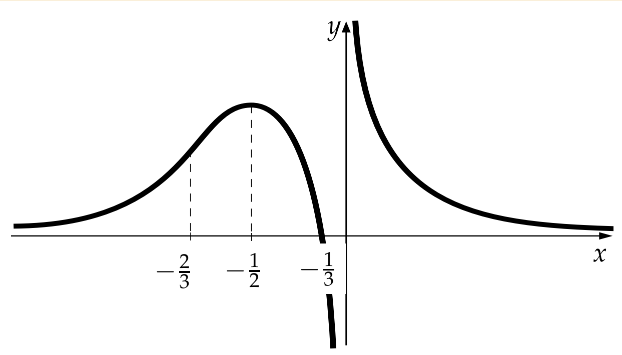


Vyšetřete chování funkce $y = \dfrac{2(x^2 - x + 1)}{(x-1)^2}$

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$,

Určíme definiční obor z podmínky

$$x - 1 \neq 0.$$

Platí

$$x \neq 1.$$

$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}\quad D(f)=\mathbb{R}\setminus\{1\},$$

$$y(0)=\frac{2(0-0+1)}{(0-1)^2}=2$$

- Určíme průsečík s osou y.
- Dosadíme $x=0$ a hledáme $y(0)$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$,

$$\frac{2(x^2 - x + 1)}{(x-1)^2} = 0$$

- Určíme průsečík s osou x.
- Dosadíme $y = 0$ a řešíme rovnici

$\boxed{y = \dfrac{2(x^2 - x + 1)}{(x-1)^2}}$ $D(f) = \mathbb{R} \setminus \{1\}, y(0) = 2,$

$$\frac{2(x^2 - x + 1)}{(x-1)^2} = 0$$

$$x^2 - x + 1 = 0$$

Čitatel musí být nula.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$$\frac{2(x^2 - x + 1)}{(x-1)^2} = 0$$
$$x^2 - x + 1 = 0$$

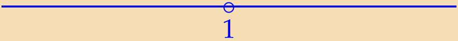
Tato kvadratická rovnice nemá řešení, protože ze vzorce

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Obdržíme záporný diskriminant.

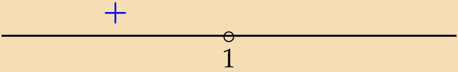
$$D = b^2 - 4ac = 2 - 4.1.1 = -2 < 0$$

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x



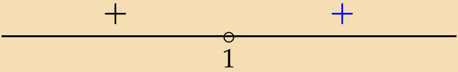
Nakreslíme osu x a bod nespojitosti $x = 1$.

$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$ $D(f)=\mathbb{R}\setminus\{1\}$, $y(0)=2$, není průsečík s osou x



Víme, že $y(0)=2>0$. Funkce je kladná na $(-\infty,1)$.

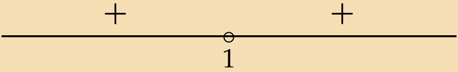
$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x



Vypočteme $y(2) = \dfrac{2(4-2+1)}{(2-1)^2} > 0$. Funkce je kladná na $(1, \infty)$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

$D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x



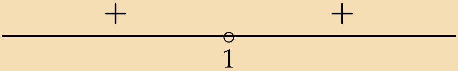
$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2}$$

$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2}$$

Určíme jednostranné limity v bodě nespojitosti

$$\boxed{y=\frac{2(x^2-x+1)}{(x-1)^2}}$$

$D(f)=\mathbb{R}\setminus\{1\}$, $y(0)=2$, není průsečík s osou x

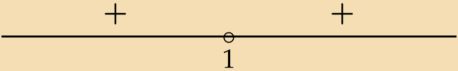


$$\lim_{x\to 1^+}\frac{2(x^2-x+1)}{(x-1)^2}=\frac{2}{0}$$

$$\lim_{x\to 1^-}\frac{2(x^2-x+1)}{(x-1)^2}=\frac{2}{0}$$

Dosadíme $x=1$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x



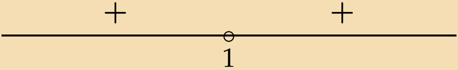
$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

Odvodíme výsledek.

$$y = \frac{2(x^2 - x + 1)}{(x-1)^2}$$

$D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x



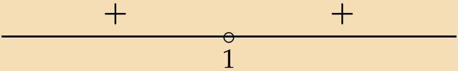
$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to \pm\infty} \frac{2(x^2 - x + 1)}{(x-1)^2}$$

Určíme limity v $\pm\infty$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x



$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

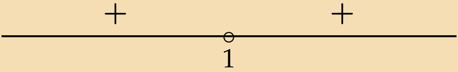
$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to \pm\infty} \frac{2(x^2 - x + 1)}{(x-1)^2} = \lim_{x\to \pm\infty} \frac{2x^2}{x^2} =$$

Uvažujeme jenom vedoucí členy.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$

$D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x



$$\lim_{x\to 1^+} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to 1^-} \frac{2(x^2 - x + 1)}{(x-1)^2} = \frac{2}{+0} = +\infty$$

$$\lim_{x\to \pm\infty} \frac{2(x^2 - x + 1)}{(x-1)^2} = \lim_{x\to \pm\infty} \frac{2x^2}{x^2} = \lim_{x\to \pm\infty} \frac{2}{1} = 2$$

Funkce má kladnou limitu v ±∞. Vodorovná přímka $y = 2$ je asymptotou ke grafu v bodech ±∞.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$$y' = 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)'$$

Vypočteme derivaci

$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$$y' = 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)'$$

$$= 2\frac{(2x-1)(x-1)^2 - (x^2 - x + 1)2(x-1)(1-0)}{((x-1)^2)^2}$$

- Užijeme vzorec pro derivaci podílu.

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}.$$

- Užijeme vzorec pro derivaci složené funkce při derivování výrazu $(x-1)^2$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$$\begin{aligned} y' &= 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)' \\ &= 2\frac{(2x-1)(x-1)^2 - (x^2 - x + 1)2(x-1)(1-0)}{((x-1)^2)^2} \\ &= 2(x-1)\frac{(2x-1)(x-1) - (x^2 - x + 1)2}{(x-1)^4} \end{aligned}$$

Vytkneme $(x-1)$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$$\begin{aligned}
y' &= 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)' \\
&= 2\frac{(2x-1)(x-1)^2 - (x^2 - x + 1)2(x-1)(1-0)}{((x-1)^2)^2} \\
&= 2(x-1)\frac{(2x-1)(x-1) - (x^2 - x + 1)2}{(x-1)^4} \\
&= 2\frac{2x^2 - 2x - x + 1 - (2x^2 - 2x + 2)}{(x-1)^3}
\end{aligned}$$

Roznásobíme závorky a zkrátíme $(x-1)$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$$\begin{aligned} y' &= 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)' \\ &= 2\frac{(2x-1)(x-1)^2 - (x^2 - x + 1)2(x-1)(1-0)}{((x-1)^2)^2} \\ &= 2(x-1)\frac{(2x-1)(x-1) - (x^2 - x + 1)2}{(x-1)^4} \\ &= 2\frac{2x^2 - 2x - x + 1 - (2x^2 - 2x + 2)}{(x-1)^3} \\ &= 2\frac{-x-1}{(x-1)^3} \end{aligned}$$

Upravíme čitatel.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$$\begin{aligned}
y' &= 2\left(\frac{x^2 - x + 1}{(x-1)^2}\right)' \\
&= 2\frac{(2x-1)(x-1)^2 - (x^2 - x + 1)2(x-1)(1-0)}{((x-1)^2)^2} \\
&= 2(x-1)\frac{(2x-1)(x-1) - (x^2 - x + 1)2}{(x-1)^4} \\
&= 2\frac{2x^2 - 2x - x + 1 - (2x^2 - 2x + 2)}{(x-1)^3} \\
&= 2\frac{-x-1}{(x-1)^3} = -2\frac{x+1}{(x-1)^3}
\end{aligned}$$

Derivace.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$$y' = -2\frac{x+1}{(x-1)^3},$$

$$-2\frac{x+1}{(x-1)^3} = 0$$

Řešíme rovnici $y' = 0$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$$y' = -2\frac{x+1}{(x-1)^3},$$

$$-2\frac{x+1}{(x-1)^3} = 0$$

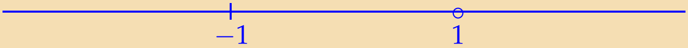
$$x + 1 = 0$$

$$x = -1$$

Čitatel musí být nula. Stacionárním bodem je tedy $x = -1$.

$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

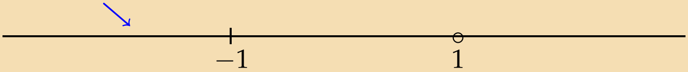
$y' = -2\frac{x+1}{(x-1)^3}$, $x_1 = -1$



zakreslíme stacionární bod a bod nespojitosti.

$\boxed{y = \dfrac{2(x^2 - x + 1)}{(x-1)^2}}$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\dfrac{x+1}{(x-1)^3}$, $x_1 = -1$

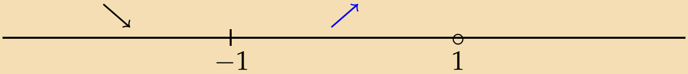


Určíme $y'(-2)$.

$$y'(-2) = -2\frac{-2+1}{(-2-1)^3} = -2\frac{\text{negative}}{\text{negative}} < 0$$

$\boxed{y = \dfrac{2(x^2 - x + 1)}{(x-1)^2}}$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\dfrac{x+1}{(x-1)^3}$, $x_1 = -1$

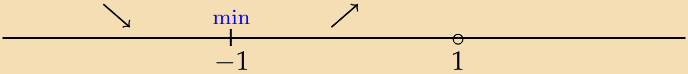


Určíme $y'(0)$.

$$y'(0) = -2\frac{0+1}{(0-1)^3} = -2\frac{\text{kladná hodnota}}{\text{záporná hodnota}} > 0$$

$\boxed{y=\dfrac{2(x^2-x+1)}{(x-1)^2}}$ $D(f)=\mathbb{R}\setminus\{1\}$, $y(0)=2$, není průsečík s osou x

$y'=-2\dfrac{x+1}{(x-1)^3}$, $x_1=-1\ldots$ lok. minimum, $y(-1)=\dfrac{3}{2}$

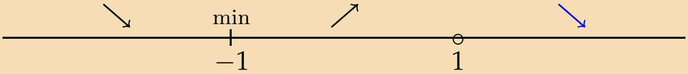


Lokální minimum pro $x=-1$. Funkční hodnota je

$$y(-1)=\frac{2((-1)^2-(-1)+1)}{(-1-1)^2}=\frac{2.3}{4}=\frac{3}{2}.$$

$\boxed{y = \dfrac{2(x^2-x+1)}{(x-1)^2}}$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\dfrac{x+1}{(x-1)^3}$, $x_1 = -1 \dots$ lok. minimum, $y(-1) = \dfrac{3}{2}$



$$\boxed{y'(2) = -2\frac{2+1}{(2-1)^3} = -2\frac{3}{1} < 0}$$

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\frac{x+1}{(x-1)^3}$, $x_1 = -1\ldots$ lok. minimum, $y(-1) = \frac{3}{2}$

$$y'' = -2\left(\frac{x+1}{(x-1)^3}\right)'$$

Vypočteme druhou derivaci.

$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$ $D(f) = \mathbb{R} \setminus \{1\}, y(0) = 2$, není průsečík s osou x

$y' = -2\frac{x+1}{(x-1)^3}, x_1 = -1 \ldots$ lok. minimum, $y(-1) = \frac{3}{2}$

$$y'' = -2\left(\frac{x+1}{(x-1)^3}\right)'$$
$$= -2\frac{1(x-1)^3 - (x+1)3(x-1)^2(1-0)}{((x-1)^3)^2}$$

- Použijeme pravidlo pro derivaci podílu.
- Jmenovatel budeme derivovat jako složenou funkci.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\frac{x+1}{(x-1)^3}$, $x_1 = -1\ldots$ lok. minimum, $y(-1) = \frac{3}{2}$

$$\begin{aligned} y'' &= -2\left(\frac{x+1}{(x-1)^3}\right)' \\ &= -2\frac{1(x-1)^3 - (x+1)3(x-1)^2(1-0)}{((x-1)^3)^2} \\ &= -2(x-1)^2\frac{(x-1) - (x+1)3}{(x-1)^6} \end{aligned}$$

Vytkneme $(x-1)^2$ v čitateli.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\frac{x+1}{(x-1)^3}$, $x_1 = -1\ldots$ lok. minimum, $y(-1) = \frac{3}{2}$

$$\begin{aligned} y'' &= -2\left(\frac{x+1}{(x-1)^3}\right)' \\ &= -2\frac{1(x-1)^3 - (x+1)3(x-1)^2(1-0)}{((x-1)^3)^2} \\ &= -2(x-1)^2\frac{(x-1)-(x+1)3}{(x-1)^6} \\ &= -2\frac{-2x-4}{(x-1)^4} \end{aligned}$$

Upravíme.

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$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\dfrac{x+1}{(x-1)^3}$, $x_1 = -1\dots$ lok. minimum, $y(-1) = \dfrac{3}{2}$

$$\begin{aligned} y'' &= -2\left(\frac{x+1}{(x-1)^3}\right)' \\ &= -2\frac{1(x-1)^3 - (x+1)3(x-1)^2(1-0)}{((x-1)^3)^2} \\ &= -2(x-1)^2\frac{(x-1) - (x+1)3}{(x-1)^6} \\ &= -2\frac{-2x-4}{(x-1)^4} = 4\frac{x+2}{(x-1)^4} \end{aligned}$$

Obdrželi jsme druhou derivaci.

$\boxed{y = \dfrac{2(x^2 - x + 1)}{(x-1)^2}}$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\dfrac{x+1}{(x-1)^3}$, $x_1 = -1 \ldots$ lok. minimum, $y(-1) = \dfrac{3}{2}$

$y'' = 4\dfrac{x+2}{(x-1)^4}$,

$$4\frac{x+2}{(x-1)^4} = 0$$

Řešíme $y'' = 0$.

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\dfrac{x+1}{(x-1)^3}$, $x_1 = -1\ldots$ lok. minimum, $y(-1) = \dfrac{3}{2}$

$y'' = 4\dfrac{x+2}{(x-1)^4}$, $x_2 = -2$

$$4\frac{x+2}{(x-1)^4} = 0$$

$$x + 2 = 0$$

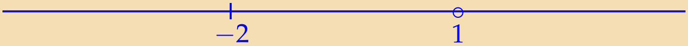
$$x = -2$$

Jediné řešení je $x = -2$.

$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\frac{x+1}{(x-1)^3}$, $x_1 = -1 \dots$ lok. minimum, $y(-1) = \frac{3}{2}$

$y'' = 4\frac{x+2}{(x-1)^4}$, $x_2 = -2$



Určíme intervaly konvexnosti a konkavity.

$\boxed{y = \dfrac{2(x^2 - x + 1)}{(x-1)^2}}$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\dfrac{x+1}{(x-1)^3}$, $x_1 = -1 \ldots$ lok. minimum, $y(-1) = \dfrac{3}{2}$

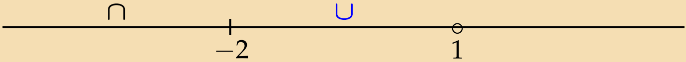
$y'' = 4\dfrac{x+2}{(x-1)^4}$, $x_2 = -2$

$$\boxed{y''(-3) = 4\frac{-3+2}{\text{kladná hodnota}} < 0}$$

$\boxed{y = \dfrac{2(x^2 - x + 1)}{(x-1)^2}}$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\dfrac{x+1}{(x-1)^3}$, $x_1 = -1\ldots$ lok. minimum, $y(-1) = \dfrac{3}{2}$

$y'' = 4\dfrac{x+2}{(x-1)^4}$, $x_2 = -2$

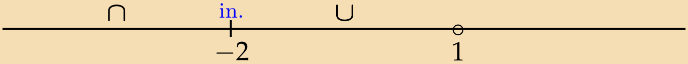


$$\boxed{y''(0) = 4\frac{0+2}{\text{kladná hodnota}} > 0}$$

$$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\dfrac{x+1}{(x-1)^3}$, $x_1 = -1 \ldots$ lok. minimum, $y(-1) = \dfrac{3}{2}$

$y'' = 4\dfrac{x+2}{(x-1)^4}$, $x_2 = -2$



Inflexní bod v bodě $x = -2$. Funkční hodnota je

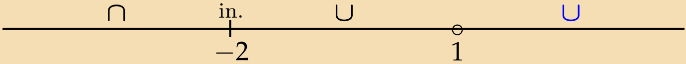
$$y(-2) = \frac{14}{9}.$$

(Vypočtěte si sami.)

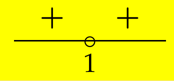
$\boxed{y = \frac{2(x^2 - x + 1)}{(x-1)^2}}$ $D(f) = \mathbb{R} \setminus \{1\}$, $y(0) = 2$, není průsečík s osou x

$y' = -2\frac{x+1}{(x-1)^3}$, $x_1 = -1\ldots$ lok. minimum, $y(-1) = \frac{3}{2}$

$y'' = 4\frac{x+2}{(x-1)^4}$, $x_2 = -2$

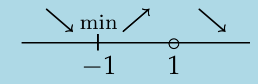


$$\boxed{y''(2) = 4\frac{2+1}{\text{kladná hodnota}} > 0}$$



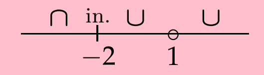
$$f(0) = 2$$

$$f(\pm\infty) = 2$$



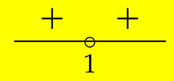
$$f(1\pm) = +\infty$$

$$f(-1) = \frac{3}{2}$$



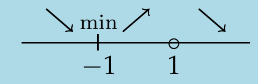
$$f(-2) = \frac{14}{9}$$

Shrneme dosavadní znalosti.



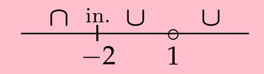
$f(0) = 2$

$f(\pm\infty) = 2$

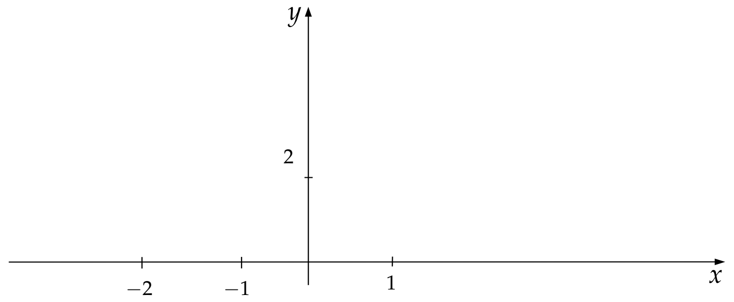


$f(1\pm) = +\infty$

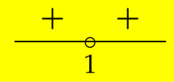
$f(-1) = \frac{3}{2}$



$f(-2) = \frac{14}{9}$

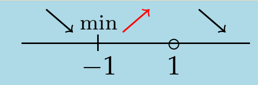


Nakreslíme souřadnou soustavu.



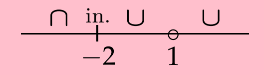
$f(0) = 2$

$f(\pm\infty) = 2$

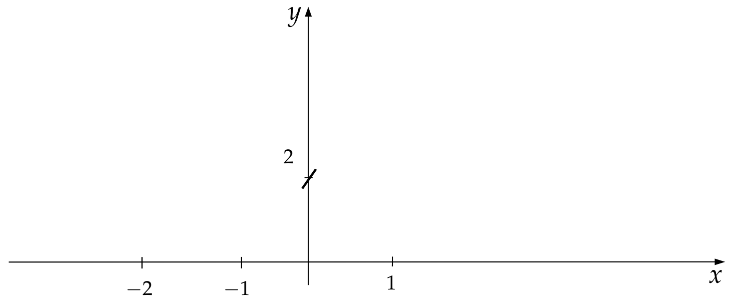


$f(1\pm) = +\infty$

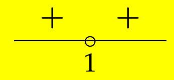
$f(-1) = \frac{3}{2}$

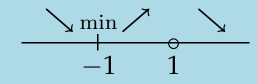


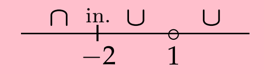
$f(-2) = \frac{14}{9}$



Vyznačíme průsečík s osou y. Funkce v tomto bodě roste.







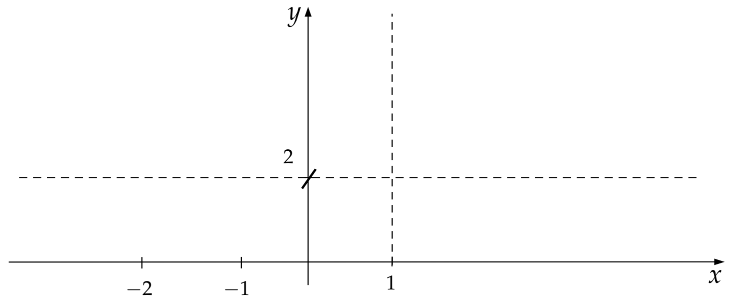
$f(0) = 2$

$f(\pm\infty) = 2$

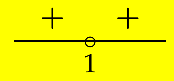
$f(1\pm) = +\infty$

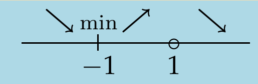
$f(-1) = \frac{3}{2}$

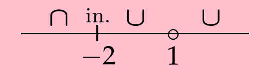
$f(-2) = \frac{14}{9}$



Nakreslíme asymptoty.







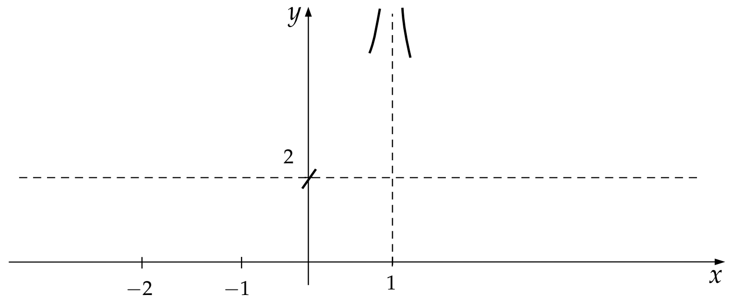
$f(0) = 2$

$f(\pm\infty) = 2$

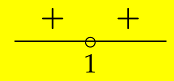
$f(1\pm) = +\infty$

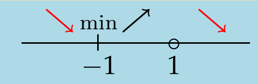
$f(-1) = \frac{3}{2}$

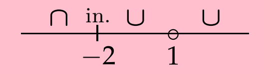
$f(-2) = \frac{14}{9}$



Nakreslíme funkci v okolí svislé asymptoty.







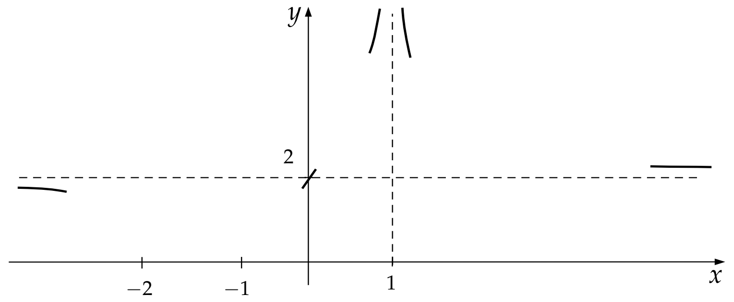
$f(0) = 2$

$f(\pm\infty) = 2$

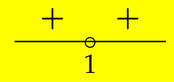
$f(1\pm) = +\infty$

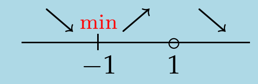
$f(-1) = \dfrac{3}{2}$

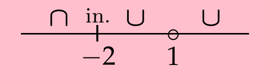
$f(-2) = \dfrac{14}{9}$



Nakreslíme funkci v okolí vodorovné asymptoty.







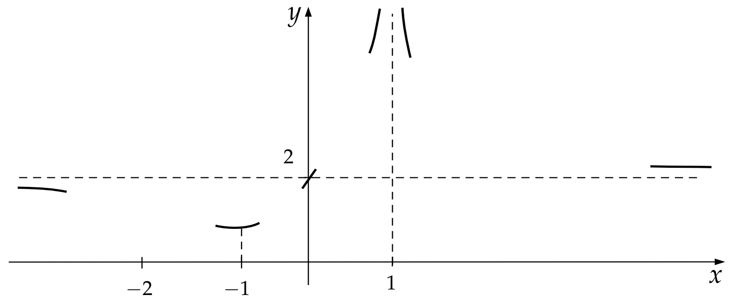
$f(0) = 2$

$f(\pm\infty) = 2$

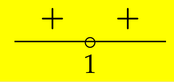
$f(1\pm) = +\infty$

$f(-1) = \frac{3}{2}$

$f(-2) = \frac{14}{9}$

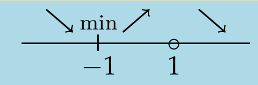


Nakreslíme lokální minimum funkce.



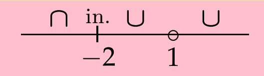
$+ \quad +$

1



↘ min ↗ ↘

$-1 \quad 1$



$\cap$ in. $\cup$ $\cup$

$-2 \quad 1$

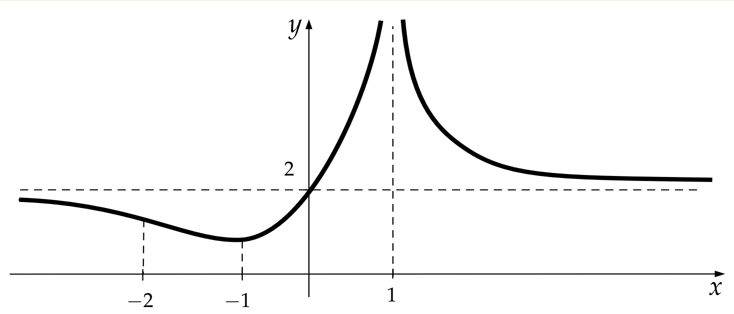
$f(0) = 2$

$f(\pm\infty) = 2$

$f(1\pm) = +\infty$

$f(-1) = \dfrac{3}{2}$

$f(-2) = \dfrac{14}{9}$



Hotovo!